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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

9500 GILMAN DRIVE
LA JOLLA, CALIFORNIA, 92093-0407

December 3, 1993

Re: Semi-annual report on ONR grant no: N00014-93-1-0698
Title: Design and Evaluation of Fault Tolerance in Opto-electronic Computing
Principle Investigator: Professor Ting-Ting Y. Lin
Addressees:
Scientific Officer, Dr. Clifford G. Lau
Administrative Grants Officer
Director, Naval Research Laboratory
Defense Technical Information Center

Dear Sirs,

This letter report, for the period of 1 June 1993 through 1 December 1993, describes the activities supported by the ASSERT grant. A student, John Lillis, is supported under this grant. John has a strong background in theory and will perform some analytical work on clustering and partitioning (for our test pipe formation) to minimize the addition of test circuits and delay. This is especially useful in constructing our processing plane. We started with two areas.

The first area of research is *Data-Flow Clustering*. In synchronous circuits (described in terms of registers and combinational nodes) it is known that the feedback loops are dominant in determining the achievable clock period for the circuit. More precisely, the maximum delay to register ratio over all loops in the circuit (called the iteration bound) is a lower bound on the clock period. Furthermore, if the combinational nodes are fine-grained in nature, the iteration bound very closely reflects the achievable clock period through retiming. However, because of size constraints it is often necessary to partition such a circuit into multiple modules (for example FPGAs). In such a situation, inter-module delay can be substantial compared to intra-module delay. Accordingly, we study the problem of clustering the nodes of the graph into different modules such that the iteration bound of the resulting circuit (where inter-module delays are taken into account when determining the delay-register ratio of a loop) is minimized. We have shown that this problem is NP-Complete even if replication of combinational nodes is allowed (ie, provided all its inputs are available, a node may appear in two different modules so as to absorb some of the inter-module delay costs of different loops; this is a popular method for minimizing delay in acyclic networks). Accordingly, we have proposed a heuristic solution to the problem and are conducting experiments to evaluate its effectiveness.

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The second area of research is network partitioning. We study the classical problem of finding the Minimum-Cut in an undirected graph, given source and sink nodes, s and t. We reformulate the problem as a continuous placement problem. We show the correspondence between the Min-Cut problem and the placement problem which is then solved iteratively using gradient methods. In addition, this method has good potential for efficient parallel implementation. We provide experimental data on the effectiveness of our approach. This work will be submitted to *Information Processing Letters*. A draft of the paper is in Appendix A.

This summarizes our research progress. Please do not hesitate to call me if this is not sufficient.

Best regards,

Ting-Ting Lin
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A Gradient Approach To The Minimum Cut Problem

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Abstract

We present a gradient approach to the classical *Min-Cut* problem for capacitated undirected graphs. This is done by showing the correspondence between a continuous minimum placement problem and the min-cut problem. We demonstrate the use of the *Successive Over Relaxation (SOR)* iterative method for solving the continuous formulation. The feasibility of this approach is demonstrated experimentally. We believe this method has potential for successful parallel implementation

1 Introduction

We present a gradient approach to the problem of finding the minimum cut in a weighted, undirected graph $G = (V, E)$ given source and sink nodes s and t, and edge capacities $c_{i,j}$ for $(i,j) \in E$ ($c_{i,j}$ is considered to be 0 if $(i,j) \notin E$). Our approach is inspired by the *Gordian-L* placement tool [1]. The paper is organized as follows:

- We will show that we can find the min-cut in a graph by minimizing the function $F(\vec{x}) = \sum_{(i,j)\in E} c_{i,j}|x_i - x_j|$ where each node i in the graph has a real-valued variable x_i associated with it except x_s and x_t which are fixed.

- We give an approximation to the above formulation by modeling it as a nonlinear resistive network.

- We show how to minimize the above approximation by way of the Successive Over-Relaxation (SOR) method to take advantage of the possible sparseness of the graph and the properties of the admitance matrix derived from the above approximation.

- Last we give experimental results of this approach.

2 Problem Formulation

Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of undirected arcs. Further, for each arc $(i, j) \in E$ let $c_{i,j}$ be the capacity of that arc.

A minimum cut problem is, given $N = (V, E)$ and two vertices $s, t \in V$, to partition V into disjoint sets V_s and V_t where $s \in V_s$ and $t \in V_t$ such that

$$C(V_s, V_t) = \sum_{i \in V_s} \sum_{j \in V_t} c_{i,j} \tag{1}$$

is minimized. Ie, the total capacity of the edges between the partitions is minimum.

Let $f(x_i, x_j) = c_{ij}|x_i - x_j|$. Now consider the continuous minimimum placement problem which minimizes the objective function

$$F(\vec{x}) = \sum_{(i,j) \in E} f(x_i, x_j) \tag{2}$$

With x_s and x_t fixed, we claim the optimal solution to the above minimization problem captures the minimum cuts of the graph. In the following, we assume that $x_s = -\frac{1}{2}$ and $x_t = \frac{1}{2}$.

Lemma 2.1 *If $\vec{x} = x_1, \ldots, x_n$ yields the minimum for expression 2, any m ($x_s < m < x_t$ and m not equal any x_i) partitions V into two sets $V_{m_t} = \{i | x_i < m\}$ and $V_{m_s} = \{i | x_i > m\}$ and this partition is a minimum cut.*

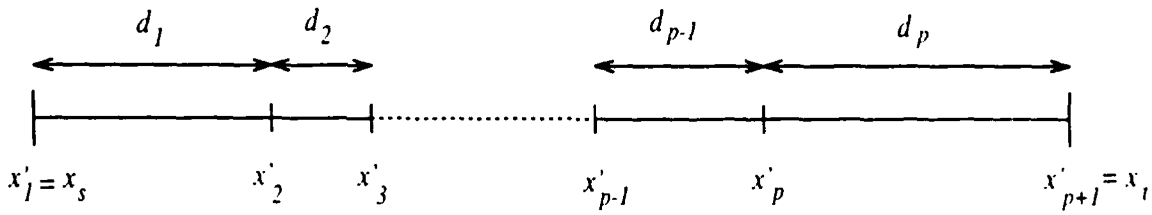

Figure 1: *Illustration of reformulation. Each segment i defines a cut between nodes on the left of the segment and the nodes on the left. The value of this cut is t_i*

Proof: Let t_{min} be the minimum cut value for graph G. Let $p + 1$ be the number of distinct values among the x_i's and let $x'_1, \cdots x'_{p+1}$ be these distinct values in ascending order (ie, there are p "segments" between x_s and x_t). We define $t_i = \sum_{x_u \le x'_i} \sum_{x_v \ge x'_{i+1}} c_{u,v}$. In other words, t_i is the total capacity of the edges crossing the i'th segment. We also define d_i as the length of the i'th segment $(x'_{i+1} - x'_i)$. This formulation is illustrated in the figure 1. Notice that $\sum_{i=1}^{p} d_i = x_t - x_s$. Clearly none of the t_i's can be less than t_{min} as this would define a cheaper cut. Therefore, we have

$$F(\vec{x}) = \sum_{i=1}^{p} t_i d_i \ge C_{min}(x_t - x_s) \tag{3}$$

Thus, in order to minimize $F(\vec{x})$, we need $t_i = C_{min}$ $\forall i$ which correspond to a cut.

3 Approximation By A Nonlinear Resistive Network

We transform our placement problem into a nonlinear resistive network. Let x_i be the voltage of node i. Given a constant ϵ, for each edge (i, j), we construct a nonlinear resistor connecting nodes i and j with a conductance, $\sigma_{i,j}$, defined as follows:

$$\sigma_{i,j} = \begin{cases} c_{i,j}/|x_i - x_j| & \text{if } |x_i - x_j| > \epsilon \\ c_{i,j}/\epsilon & \text{if } |x_i - x_j| \leq \epsilon \end{cases} \tag{4}$$

Figure 2 graphically depicts the conductance σ.

The current flowing from node i to node j is equal to the product of the voltage difference $|x_i - x_j|$ and the conductance $\sigma_{i,j}$, i.e.

$$(x_i - x_j)\sigma_{i,j} = \begin{cases} \delta_{i,j} c_{i,j} & \text{if } |x_i - x_j| > \epsilon \\ c_{i,j}(x_i - x_j)/\epsilon & \text{if } |x_i - x_j| \leq \epsilon \end{cases} \tag{5}$$

where $\delta_{i,j} = 1$ if $x_i > x_j$; $\delta_{i,j} = -1$ if $x_i < x_j$. Equation (5) is an approximation of the partial derivative of our objective function F with respect to x_i. Consequently, Kirchhoff's current law [3] in the nonlinear resistive network corresponds to the necessary condition of a zero gradient. We claim the analogy between our problem and the above resistive network.

Theorem 3.1 *Given a constant Υ, ther exists an $\epsilon = \frac{\Upsilon}{|E| max_{i,j} c_{i,j}}$ for the conductance equation (4) such that the voltage solution of the trnsformed nonlinear resistive network is an approximate solution of F with an error bound Υ on the value of F.*

Proof: To prove the theorem, we introduce the function,

$$h(x) = \begin{cases} 2|x| - \epsilon & \text{if } |x| > \epsilon \\ x^2/\epsilon & \text{if } |x| \leq \epsilon \end{cases}$$

and a potential function approximating the objective function F,

$$\Psi(\vec{x}) = \sum_{(i,j)} v(x_i, x_j) \tag{6}$$

where

$$v(x_i, x_j) = \frac{1}{2} c_{i,j} h(x_i - x_j) \tag{7}$$

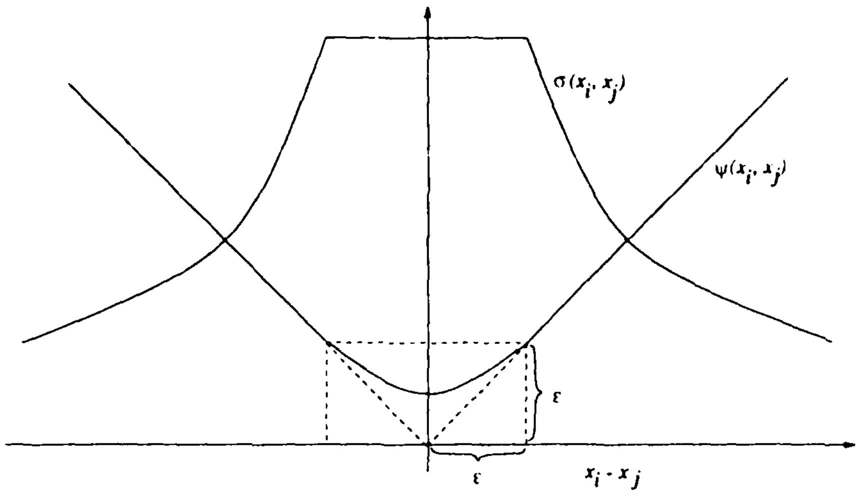

Figure 2: *Illustration of* $\psi(x_i, x_j)$ *and* $\sigma(x_i, x_j)$. *Here,* $\epsilon = \frac{1}{2}$ *and* $c_{ij} = 1$.

Figure 2 graphically depicts ψ.

We prove the theorem by proving the three following lemmas (proofs appear in the appendix):

Lemma 3.1 *The solution of the nonlinear resistive network (5) derives the minimum* $\Psi(x)$.

Lemma 3.2 *Given a positive number* γ, $|F(x) - \Psi(x)| \leq \gamma$ *holds if* $\epsilon = \frac{\gamma}{|E| max_{i,j} c_{ij}}$, *where* $|E|$ *is the number of edges in the graph.*

Lemma 3.3 *The inequality* $|F(x) - \Psi(x)| \leq \gamma$ *holds if* $\epsilon \leq \frac{\gamma}{|E| max_{i,j} c_{ij}}$.

Therefore from the above three lemmas, we have proved Theorem 3.1.

0. Set $\epsilon = \epsilon$-init

1. Set initial x vector

2. Calculate σ_{ij} according to current x vector

3. Solve $G_{11}x_1 = -G_{12}x_2$ as a linear resistive network.

4. If Change in Objective Function is less than Δ

$\epsilon = \epsilon \cdot \epsilon$-factor

5. Repeat steps 2, 3 and 4 until convergence

Figure 3: *Iterative algorithm for solving the continuous placement problem. Step 3 is performed by the* SOR *iterative method. In our experiments, Δ is a function of ϵ and the current objective Ψ. See Experimental Results section for details.*

4 Piecewise Linear Algorithm

We propose to solve our resistive network by starting at an intitial vector $\vec{x}$. Next, we treat the circuit as a linear resistive network, and find the solution with the Successive Over Relaxation (SOR) method [2] We then update the current solution $\vec{x}$ and repeat the process until the solution converges (see figure 3). In the figure, x_1 represents the variable node positions and x_2 denotes the fixed node positions (x_s and x_t). G_{11} and G_{12} denote the submatrices of admittance matrix derived from σ corresponding to vectors x_1 and x_2. Hence, the problem reduces to solving $G_{11}x_1 = -G_{12}x_2$.

In addition to σ (4), we introduce the following function:

$$g(x) = \begin{cases} 1/|x| & \text{if } |x| > \epsilon \\ 1/\epsilon & \text{if } |x| \le \epsilon \end{cases} \tag{8}$$

In the following, let $\bar{x}$ and $\tilde{x}$ be the vector at the kth and $k+1$st iterations respectively. In

addition, for notational convenience, let

$$\bar{x}_{ij} = \bar{x}_i - \bar{x}_j \tag{9}$$

$$\tilde{x}_{ij} = \tilde{x}_i - \tilde{x}_j \tag{10}$$

We introduce

$$P(\bar{x}, \tilde{x}) = \sum_{(i,j)} c_{ij} g(\bar{x}_{ij}) \tilde{x}_{ij}^{2} \tag{11}$$

$$P(\bar{x}, \bar{x}) = \sum_{(i,j)} c_{ij} g(\bar{x}_{ij}) \bar{x}_{ij}^{2} \tag{12}$$

Theorem 4.1 *For each iteration of steps 2, 3 and 4, the potential function $\Psi(x)$ is strictly decreasing.*

We prove $\Psi(\bar{x}) > \Psi(\tilde{x})$ by proving the two following lemmas (proofs appear in the appendix).

Lemma 4.1 *Before the solution converges, $P(\bar{x}, \bar{x}) > P(\bar{x}, \tilde{x})$*

Lemma 4.2 *Before the solution converges,*

$$2\Psi(\bar{x}) - P(\bar{x}, \bar{x}) \geq 2\Psi(\tilde{x}) - P(\bar{x}, \tilde{x}) \tag{13}$$

We conclude the proof of the Theorem 4.1 from the preceding lemmas.

5 Experimental Results

In this section, we present experimental results of our approach. Test graphs were gotten from the *washington.c* (developed by Richard Anderson and students at the University of Washington) program at DIMACS. It generates a variety of graphs for the directed version of the min-cut problem. We use a post process to convert these graphs into undirected graphs.

In our experiments, we fixed $x_s = -100.0$, $x_t = 100.0$, initial $\epsilon = 200.0$, ϵ-factor$= 0.80$ and $\Delta = F(x) \cdot \epsilon \cdot 10^{-3}$. In addition, we set ω, the relaxation parameter of *SOR* to be 1.9.

$\|V\|$	$\|E\|$	SOR Executions	Total Iterations	CPU time
102	590	314	8177	5.6
227	1340	423	11538	18.7
401	2380	656	12436	40.6

Table 1: *Experimental results*

6 Concluding Remarks

We have demonstrated the theoretical foundations and experimental feasibility of a gradient approach to the Min-Cut problem. Future work may include detailed analysis of the convergence rate, more exhaustive evaluation of algorithmic parameters and parallel implementation.

References

[1] G. Sigl, K. Doll, F. M. Johannes, "Analytical Placement: A Linear or a Quadratic Objective Function?," *Proc. of Int. Conf. on Computer-Aided Design.*, pp. 427-432, June 1991.

[2] G.H. Golub and C.F. Van Loan, *Matrix Computations*, Baltimore, MD: Johns Hopkins University Press, 1983.

[3] C.A. Desoer and E.S. Kuh, *Basic Circuit Theory*, New York, NY: McGraw-Hill, 1969.

7 Appendix

We present proofs of lemmas stated in the body of the paper.

Lemma 3.1 *The solution of the nonlinear resistive network (5) derives the minimum* $\Psi(x)$.

Proof: The lemma follows the reference [3], pp. 776. Since equation (6) is continuous in its first derivative, its gradient is equal to zero at a minimum solution. We can derive that the

KCL equation for the nonlinear resistive network is equivalent to the necessary condition that the gradient of (6) is zero, which proves the lemma.

Lemma 3.2 *Given a positive number* γ, $|F(x) - \Psi(x)| \leq \gamma$ *holds if* $\epsilon = \frac{\gamma}{|E| max_{i,j} c_{i,j}}$, *where* $|E|$ *is the number of edges in the graph.*

Proof: Consider the following two cases:

(i) $|x_i - x_j| > \epsilon$

From equations (2) and (6), we know that

$$|f(x_i, x_j) - \psi(x_i, x_j)| = c_{i,j}\epsilon/2 \tag{14}$$

(ii) $|x_i - x_j| \leq \epsilon$

$$|f(x_i, x_j) - \psi(x_i, x_j)| = c_{i,j}||x_i - x_j| - \frac{(x_i - x_j)^2}{2\epsilon}| \leq c_{i,j}\epsilon \tag{15}$$

Therefore, given $\epsilon = \frac{\gamma}{|E| \max_{i,j} c_{ij}}$ we have

$$|F(x) - \Psi(x)| \leq \sum_{(i,j)} |f(x_i - x_j) - \psi(x_i, x_j)| \leq \gamma \tag{16}$$

Lemma 3.3 *The inequality* $|F(x) - \Psi(x)| \leq \gamma$ *holds if* $\epsilon \leq \frac{\gamma}{|E| max_{ij} c_{ij}}$.

Proof: In the following, let $\hat{x}$ and $\tilde{x}$ be the respective global minima of $F(x)$ and $\Psi(x)$, I.e.

$$F(\hat{x}) \leq F(x) \qquad \forall x \tag{17}$$

$$\Psi(\tilde{x}) \leq \Psi(x) \qquad \forall x \tag{18}$$

We can derive

$$F(\hat{x}) - \Psi(\tilde{x}) = F(\hat{x}) - F(\tilde{x}) + F(\tilde{x}) - \Psi(\tilde{x}) \tag{19}$$

Since $F(\hat{x}) - F(\tilde{x}) \leq 0$ by definition, from the previous lemma, we have

$$F(\hat{x}) - \Psi(\tilde{x}) \leq F(\hat{x}) - F(\tilde{x}) + \gamma \leq \gamma \tag{20}$$

Similarly, we can derive

$$F(\bar{x}) - \Psi(\bar{x}) = F(x) - \Psi(x) + \Psi(x) - \Psi(\bar{x}) \geq -\gamma. \tag{21}$$

Thus from inequalities (20) and (21), we conclude that

$$|F(x) - \Psi(\bar{x})| \leq \gamma \tag{22}$$

Lemma 4.1 *Before the solution converges,* $P(x, x) > P(x, \bar{x})$

Proof: We use the property of the linear resistive network [3], pp. 770, that the minimum power disipation of the network occurs at the only solution of the network equations. Therefore, before the solution converges, the solution of $G_{11}x_1 = -G_{12}x_2$ always derives the voltage that reduces the power dissipation of the linear resistive network, i.e. $P(x, x) > P(x, \bar{x})$

Lemma 4.2 *Before the solution converges,*

$$2\Psi(\bar{x}) - P(x, x) \geq 2\Psi(\bar{x}) - P(x, \bar{x}) \tag{23}$$

Proof: We can derive that

$$2\Psi(\bar{x}) - P(\bar{x}, \bar{x}) = \sum_{(i,j)} c_{ij}h(\bar{x}_{ij}) - c_{ij}g(x_{ij})x_{ij}^2 = \sum_{(i,j)} c_{ij}[h(x_{ij}) - g(x_{ij})x_{ij}^2] \tag{24}$$

$$2\Psi(\bar{x}) - P(x, \bar{x}) = \sum_{(i,j)} c_{ij}h(\bar{x}_{ij}) - c_{ij}g(\bar{x}_{ij})\bar{x}_{ij}^2 = \sum_{(i,j)} c_{ij}[h(x_{ij}) - g(x_{ij})\bar{x}_{ij}^2] \tag{25}$$

We now show that in the following four cases,

$$h(\bar{x}_{ij}) - g(\bar{x}_{ij})\bar{x}_{ij}^2 \geq h(\bar{x}_{ij}) - g(x_{ij})\bar{x}_{ij}^2 \qquad \forall i, j \tag{26}$$

case (i) $|\bar{x}_{ij}| \leq \epsilon, |\bar{x}_{ij}| \leq \epsilon$

$$h(\bar{x}_{ij}) - g(\bar{x}_{ij})x_{ij}^2 = \frac{\bar{x}_{ij}^2}{\epsilon} - \frac{1}{\epsilon}x_{ij}^2 = 0 \tag{27}$$

$$h(\bar{x}_{ij}) - g(x_{ij})x_{ij}^2 = \frac{\bar{x}_{ij}^2}{\epsilon} - \frac{1}{\epsilon}\bar{x}_{ij}^2 = 0 \tag{28}$$

case (ii) $|\bar{x_{ij}}| \leq \epsilon, |\bar{x_{ij}} > \epsilon$

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})x_{ij}^2 = \frac{\bar{x_{ij}}^2}{\epsilon} - \frac{1}{\epsilon}\bar{x_{ij}}^2 = 0 \tag{29}$$

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})\bar{x_{ij}}^2 = 2|\bar{x_{ij}}| - \epsilon - \frac{1}{\epsilon}\bar{x_{ij}}^2 = -\frac{(|\bar{x_{ij}}| - \epsilon)^2}{\epsilon} < 0 \tag{30}$$

case (iii) $|\bar{x_{ij}}| > \epsilon, |\bar{x_{ij}} \leq \epsilon$

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})x_{ij}^2 = 2|\bar{x_{ij}}| - \epsilon - \frac{1}{|\bar{x_{ij}}|}\bar{x_{ij}}^2 = |\bar{x_{ij}}| - \epsilon \tag{31}$$

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})\bar{x_{ij}}^2 = \frac{\bar{x_{ij}}^2}{\epsilon} - \frac{1}{|\bar{x_{ij}}|}\bar{x_{ij}}^2 = \frac{\bar{x_{ij}}^2(|\bar{x_{ij}}| - \epsilon)}{\epsilon|\bar{x_{ij}}|} \leq |\bar{x_{ij}}| - \epsilon \tag{32}$$

case (iv) $|\bar{x_{ij}}| > \epsilon, |\bar{x_{ij}}| > \epsilon$

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})x_{ij}^2 = |\bar{x_{ij}}| - \epsilon \tag{33}$$

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})\bar{x_{ij}}^2 = 2|\bar{x_{ij}}| - \epsilon - \frac{1}{\epsilon}\bar{x_{ij}}^2 \tag{34}$$

Since

$$\bar{x_{ij}}^2 \geq 2|\bar{x_{ij}}||\bar{x_{ij}}| - \bar{x_{ij}}^2 \tag{35}$$

we have

$$2|\bar{x_{ij}}| - \epsilon - \frac{1}{|\bar{x_{ij}}|}\bar{x_{ij}}^2 \leq |\bar{x_{ij}}| - \epsilon \tag{36}$$

Thus, we can derive

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})\bar{x_{ij}}^2 \leq |\bar{x_{ij}}| - \epsilon \tag{37}$$

We conclude from cases (i) - (iv) that

$$h(\bar{x_{ij}}) - g(\bar{x_{ij}})\bar{x_{ij}}^2 \geq h(\bar{x_{ij}}) - g(\bar{x_{ij}})\bar{x_{ij}}^2 \tag{38}$$